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On Exact Superpotentials, Free Energies and Matrix Models

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Abstract

We discuss exact results for the full nonperturbative effective superpotentials of four dimensional $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theories with additional chiral superfield in the adjoint representation and the free energies of the related zero dimensional bosonic matrix models with polynomial potentials in the planar limit using the Dijkgraaf-Vafa matrix model prescription and integrating in and out. The exact effective superpotentials are produced including the leading Veneziano-Yankielowicz term directly from the matrix models. We also discuss how to use integrating in and out as a tool to do random matrix integrals in the large N limit.

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1 Introduction

The exact nonperturbative effective superpotential of $\mathcal{N} = 2$ supersymmetric gauge theories with all instanton corrections is completely determined with a knowledge of the perturbative UV physics, the singularities in moduli space and the monodromies around the singularities. [1] The field content of a pure $\mathcal{N} = 2$ gauge theory is a vector field A_μ, two Weyl fermions λ_α and ψ_α, and a complex scalar A. In $\mathcal{N} = 1$ language, these are a combination of a field strength chiral superfield W_α containing λ_α and A_μ, and a scalar chiral superfield Φ containing A and ψ_α all transforming in the adjoint representation. Suppose this pure $\mathcal{N} = 2$ theory is perturbed by a tree level superpotential including a mass term for Φ. When one integrates out Φ, the low energy theory below the mass of Φ reduces to $\mathcal{N} = 1$. About a decade ago, various examples of exact superpotentials of $\mathcal{N} = 1$ supersymmetric gauge theories were obtained making use of holomorphy and symmetry arguments. See [4] for a review. However, for such general tree level perturbations, combinations of parameters which are not protected by symmetries appear. These parameters can come in the effective superpotential with any degree. Dijkgraaf and Vafa [2] found that the effective superpotentials of $\mathcal{N} = 1$ theories obtained by such deformations of $\mathcal{N} = 2$ could be computed by using matrix models.

Consider pure $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory. The $SU(N)$ subgroup confines in the infrared and using the one loop running of the gauge coupling coefficient the Lagrangian that describes the theory at the cut off can be written as

$$\mathcal{L} = \int d^2\theta \, 3N \log(\frac{\Lambda_1}{\Lambda_{1c}}) S + c.c. \,, \tag{1.1}$$

where S is the glueball superfield defined in terms of the gauge chiral superfield as

$$S = -\frac{1}{32\pi^2} \mathrm{Tr}\, W^\alpha W_\alpha \,, \tag{1.2}$$

Λ_1 is the scale of the $\mathcal{N} = 1$ $SU(N)$ gauge theory and Λ_{1c} is the UV cutoff. The point is that from (1.1) one sees that $3N \log(\Lambda_1)$ linearly couples to S and acts as its source. On the other hand, gaugino condensation gives the nonperturbative effective superpotential

$$W_{\text{eff}} = N\Lambda_1^3. \tag{1.3}$$

Note that we have suppressed the N phases $e^{2\pi i k/N}$, $k = 0$ to $N-1$, in (1.3) due to the breaking of the Z_{2N} R symmetry down to Z_2 by gaugino condensation. We can integrate in S to (1.3) with $3N \log(\Lambda_1)$ as its source and calculate the glueball superpotential while still keeping Λ_1 as a parameter in the theory by introducing an auxiliary field A and minimizing

$$W = NA^3 - 3NS \log(\frac{A}{\Lambda_1}) \tag{1.4}$$

with A which gives the Veneziano-Yankielowicz effective superpotential [7]

$$W_{\text{eff}} = NS - NS \log(\frac{S}{\Lambda_1^3}). \tag{1.5}$$

Integrating out S in (1.4) simply gives back the original nonperturbative superpotential (1.3).

Suppose one adds a chiral superfield Φ in the adjoint representation with a tree level superpotential

$$W_{\text{tree}} = \sum_{p=1}^{n} \frac{g_{p+1}}{p+1} \text{Tr}\, \Phi^{p+1}. \tag{1.6}$$

to the $\mathcal{N} = 1$ $U(N)$ gauge theory. For instance, for the specific case of a cubic tree level superpotential with $n = 2$ in (1.6) and the gauge symmetry in the low energy theory unbroken, there is a combination of parameters $g_3^2 S/g_2^3$ that has no charge under all symmetries. This parameter can appear with any power and the instanton corrections to the effective nonperturbative superpotential can be written as $\sum_{k=1}^{\infty} c_k (g_3^2 S/g_2^3)^k S$ where c_k are constants. Our interest is to find exact analytic expressions for the glueball effective superpotential whose series expansion produces all terms including the leading Veneziano-Yankielowicz and the instanton corrections and to compute the free energies of the corresponding zero dimensional bosonic matrix models using integrating in [3] and out techniques. The relation between the Dijkgraaf-Vafa matrix model prescription and the integrating in method for $U(N)$ gauge theories was discussed in [5, 6] and our approach on the integrating in side here will be along similar lines.

For completeness we will start with a very brief review of a zero dimensional $U(N)$ bosonic matrix model as originally developed in [8] and reviewed in [9]. We will then compute the complete exact nonperturbative superpotentials for a quadratic, cubic and quartic tree level superpotentials using the Dijkgraaf-Vafa matrix model prescription. The free energies in the matrix model for cubic and quartic potentials were originally computed in [8] and the corresponding instanton corrections to effective superpotentials were computed in [10, 11] using the Dijkgraaf-Vafa prescription. However, the leading Veneziano-Yankielowicz term was added by hand. Here we will point out the proper normalization of the partition function and scheme that produces the exact effective superpotential including the Veneziano-Yankielowicz term directly from the matrix model. We will then continue with computing the full exact effective superpotential for quadratic, cubic and quartic tree level deformations using integrating in and out. The results from the two approaches exactly agree to all order including the leading Veneziano-Yankielowicz term and the instanton corrections. We will also provide a scheme for using integrating in and out to do random matrix integrals. For the case of one-cut $U(N)$ bosonic matrix models we will discuss in this note, integrating in and out combined with the Dijkgraaf-Vafa matrix model prescription provides a very simple tool to do random matrix integrals in the planar limit for any tree level polynomial potential.

2 Matrix model

Consider a zero dimensional $U(N)$ bosonic matrix model with a potential given by (1.6). Φ is an $N \times N$ matrix. Extremizing the potential generically gives n distinct values a_I of Φ, where $I = 1$ to n. If Φ is taken to have classical eigenvalues a_I each with degeneracy N_I such that $N = \sum_{I=1}^{n} N_I$, then the gauge symmetry in the low energy theory is broken to $\prod_{I=1}^{n} U(N_I)$. Our interest in this note is the case where all a_I are equal and the gauge symmetry is preserved. We will take $a_I = 0$.

Now consider the partition function

$$Z = \frac{1}{\mathrm{Vol}(U(N))} \int d\Phi \, e^{-\frac{1}{g_s} W_{\rm tree}}. \tag{2.1}$$

The large N (or small g_s) limit is done with $N \to \infty$ and $g_s \to 0$ where

$$S \equiv N g_s \tag{2.2}$$

is fixed. The perturbative expansion of (2.1) leads to a sum of Feynman diagrams over Riemann surfaces. In the large N limit only the leading order sum over a genus zero planar surface survives and the partition function becomes

$$Z = e^{-\frac{\mathcal{F}_0}{g_s^2}}, \tag{2.3}$$

where $\mathcal{F}_0$ is the free energy in the planar limit.

In the large N limit, the integral is done using the standard matrix model technology where one starts with transforming the integral from Φ to a set of eigenvalues λ_i of Φ. The matrix integral (2.1) when transformed to integral over eigenvalues λ_i becomes

$$Z = \int \prod_i d\lambda_i \, e^{-\frac{1}{g_s}\sum_i W(\lambda_i) + 2\sum_{i<j} \log|\lambda_i - \lambda_j|}, \tag{2.4}$$

where we write the tree level superpotential in (1.6) as $W_{\rm tree}(\Phi) = \mathrm{Tr}\, W(\Phi)$. The indices i and j in this section run over 1 to N. The second term in the exponent of the integrand in (2.4) comes from the Jacobian in the transformation of the measure of the integral from the independent components of Φ to the eigenvalues. This term gives rise to a repulsive interaction between the eigenvalues resulting in distributions of eigenvalues around each extremum point. In the case where the original gauge symmetry is preserved in the low energy theory and the extremum point is taken to be at the origin, all the eigenvalues are distributed around the origin. The equation of motion of the eigenvalues is

$$\frac{1}{g_s} W'(\lambda_i) - 2\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 0. \tag{2.5}$$

Equation (2.5) is difficult to solve directly in practice and one introduces the resolvent

$$w(x) \equiv \frac{1}{N} \sum_i \frac{1}{\lambda_i - x}, \tag{2.6}$$

where x is complex. Note that $w(x)$ has the large x asymptotic behavior

$$w(x) \to -1/x. \tag{2.7}$$

The equation of motion of the eigenvalues in the large N limit gives a quadratic equation for the resolvent,

$$w(x)^2 + \frac{1}{S} W'(x) w(x) + \frac{1}{4S^2} f(x) = 0, \tag{2.8}$$

where

$$f(x) = \frac{4S}{N} \sum_i \frac{W'(x) - W'(\lambda_i)}{x - \lambda_i} \tag{2.9}$$

is a polynomial function of degree $n-1$. (2.8) has the solution

$$w(x) = -\frac{1}{2S}\Big(W'(x) \pm \sqrt{W'(x)^2 - f(x)}\Big). \tag{2.10}$$

Because $W'(x)$ and $f(x)$ are polynomial functions, any singular behavior of $w(x)$ comes from the square root in the second term in (2.10). Note that $W'(x)^2 - f(x)$ is a polynomial of degree $2n$ and thus $w(x)$ has in general the same number of branch points. This singularity structure and the asymptotic large x behavior of $w(x)$ are enough to completely determine $w(x)$ in the one branch cut case where $n-1$ roots of $W'(x)^2 - f(x)$ come in pair. In the large N limit, one defines the eigenvalue density,

$$\rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda_i - x), \tag{2.11}$$

such that $\int \rho(\lambda) d\lambda = 1$. The point is that $w(x)$ and $\rho(x)$ are related,

$$w(x) = \int \frac{\rho(\lambda) d\lambda}{\lambda - x}. \tag{2.12}$$

Therefore, once $w(x)$ is obtained, we can invert (2.11) to compute the eigenvalue density,

$$\rho(\lambda) = \frac{1}{2\pi i}\Big(w(\lambda + i0) - w(\lambda - i0)\Big). \tag{2.13}$$

With $\rho(\lambda)$ in hand, the free energy in the large N limit follows from (2.3) and (2.4),

$$\mathcal{F}_0 = S \int d\lambda \rho(\lambda) W(\lambda) - S^2 \int\int d\lambda d\mu\, \rho(\lambda)\rho(\mu) \log|\lambda - \mu|. \tag{2.14}$$

Using the equations of motion given by (2.5) in the large N limit, (2.14) can be written in a more convenient form,

$$\mathcal{F}_0 = \frac{1}{2} S \int d\lambda\, \rho(\lambda) W(\lambda) - S^2 \int d\lambda\, \rho(\lambda) \log|\lambda|. \tag{2.15}$$

Following Dijkgraaf-Vafa [2], the effective superpotential is given by

$$W_{\text{eff}} = N \frac{\partial \mathcal{F}_0}{\partial S} + NS \log(\Lambda^2), \tag{2.16}$$

where $N \log(\Lambda^2)$ is a source for S and Λ is the nonperturbative scale in the $\mathcal{N} = 2$ gauge theory.

The important point and scheme we want to point out is that with the proper normalization of the partition function as given by (2.1) and the planar limit of the free energy that follows from (2.4), the exact effective superpotential is produced using (2.16) without a need for adding the Veneziano-Yankielowicz term by hand. We will explicitly see in subsection (2.1) that the exact Veneziano-Yankielowicz superpotential follows from a quadratic tree level potential with this scheme. We will also derive the appropriate free energies in the planar limit for cubic and quartic tree level superpotentials and show that the full exact effective superpotentials including both the Veneziano-Yankielowicz term and all instanton correction are produced.

2.1 Quadratic potential

Consider a quadratic mass term with $n = 1$ and $g_2 = m$ in $W_{\rm tree}$ given by (1.6). Now $W(x) = mx^2/2$ has only one extremum point and it occurs at the origin. The most general form of $w(x)$ follows from (2.7) and (2.10),

$$w(x) = -\frac{1}{2S}(mx - m\sqrt{x^2 - a^2}), \tag{2.17}$$

where $a = 2\sqrt{S/m}$. The Wigner semicircle density distribution of the eigenvalues follows from (2.17) in (2.13),

$$\rho(\lambda) = \frac{m}{2\pi S}\sqrt{a^2 - \lambda^2}, \tag{2.18}$$

for $\lambda \in [-a, a]$. Note that because $W_{\rm tree}$ is even, the expectation value of Φ vanishes and $U(N)$ automatically reduces to $SU(N)$. Using (2.18) and $W(\lambda) = m\lambda^2/2$ in (2.15), we obtain the free energy

$$\mathcal{F}_0 = \frac{3}{4}S^2 - \frac{1}{2}S^2 \log(\frac{S}{m}). \tag{2.19}$$

The effective superpotential follows from (2.19) in (2.16),

$$W_{\rm eff} = NS - NS\log(\frac{S}{\Lambda_1^3}), \tag{2.20}$$

where $\Lambda_1 = (\Lambda^2 m)^{1/3}$ is the scale of the lower energy pure $\mathcal{N} = 1$ theory related to the scale Λ of the higher energy $\mathcal{N} = 2$ theory by threshold matching of the gauge coupling running at the energy scale m. Now (2.20) is exactly the Veneziano-Yankielowicz superpotential. It was discussed in [2, 12] that the volume of the $U(N)$ gauge group gives a contribution like $\frac{1}{2}S^2\log(S)$ to the free energy in the planar limit. For a recent discussion on this point and on adding the Veneziano-Yankielowicz term by hand, see [11]. What we have shown here is that with the proper normalization and scheme discussed earlier in this section, the appropriate free energy and the exact Veneziano-Yankielowicz superpotential are obtained. There is no need for adding the Veneziano-Yankielowicz term by hand in computing the exact effective superpotential for any tree level superpotential deformation using this scheme.

2.2 Cubic potential

Next consider the cubic tree level superpotential,

$$W_{\rm tree} = \frac{1}{2}m\,{\rm Tr}\,(\Phi^2) + \frac{1}{3}g\,{\rm Tr}\,(\Phi^3). \tag{2.21}$$

Analytic expressions for the free energy of this model and the quartic potential in the next subsection were first obtained in [8] and the instanton corrections in the effective superpotential using the Dijkgraaf-Vafa prescription have been computed in [10] with the Veneziano-Yankielowicz term added by hand. Here we will compute the effective superpotential including the Veneziano-Yankielowicz term directly. Now there are two critical points, one at $x = 0$ and the other at $x = -m/g$. For the case where the $U(N)$ gauge symmetry is unbroken, we need to consider only one critical point and

we choose the one at the origin. In the quantum theory, all eignevalues are then distributed in an interval $[a,\, b]$ enclosing the extremum point at $x=0$. The resolvent is then completely determined noting that $W'(x)^2 - f(x)$ in (2.10) has one double root and using the asymptotic behavior (2.7), and it is given by

$$w(x) = -\frac{1}{2S}\Big(gx^2 + mx - (gx + \frac{1}{2}(a+b)g + m)\sqrt{(x-a)(x-b)}\Big), \qquad (2.22)$$

where

$$\frac{1}{4}g(a+b)^2 + \frac{1}{2}m(a+b) + \frac{1}{8}g(b-a)^2 = 0 \qquad (2.23)$$

and

$$(b-a)^2[(a+b)g+m] - 16S = 0. \qquad (2.24)$$

Combining the two conditions (2.23) and (2.24) and defining

$$\sigma = \frac{g}{2m}(a+b), \qquad (2.25)$$

we obtain

$$\sigma(1+\sigma)(1+2\sigma) + \frac{2g^2}{m^3}S = 0 \qquad (2.26)$$

with solution

$$\sigma = -\frac{1}{2} + \frac{1}{2\sqrt{3}}(A + \frac{1}{A}), \qquad (2.27)$$

where

$$A = \Big(\sqrt{432\frac{g^4S^2}{m^6} - 1} - 12\sqrt{3}\frac{g^2S}{m^3}\Big)^{1/3}. \qquad (2.28)$$

We have chosen the solution σ to the cubic equation (2.26) such that a and b are real and the eigenvalues are distributed on the real axis of x for real g^2S/m^3. The eigenvalue density follows from (2.22) and (2.13), and writing it in terms of σ and making a change of variable,

$$\rho(y) = \frac{1}{2\pi S}\Big(gy + (1+2\sigma)m\Big)\sqrt{y_0^2 - y^2}, \qquad (2.29)$$

where $y = \lambda - m\sigma/g$ and $y_0 = 2\sqrt{S/((1+2\sigma)m)}$. The eigenvalues are distributed in the range $\lambda \in [a,\, b]$ or equivalently in $y \in [-y_0,\, y_0]$.

The free energy can then be calculated using (2.29) in (2.15) which gives

$$\mathcal{F}_0 = -\frac{1}{2}S^2\log(\frac{S}{m(1+2\sigma)}) + S^2\frac{24\sigma^3 + 48\sigma^2 + 37\sigma + 9}{12(1+\sigma)(1+2\sigma)^2}. \qquad (2.30)$$

Using (2.30) in (2.16) and noting

$$\frac{\partial\sigma}{\partial S} = \frac{\sigma(1+\sigma)(1+2\sigma)}{6\sigma^2 + 6\sigma + 1} \qquad (2.31)$$

from (2.26), we obtain

$$W_{\rm eff} = \frac{1}{3}NS\Big(2 + \frac{2}{1+2\sigma} - \frac{1}{1+\sigma}\Big) - NS\log(\frac{S}{(1+2\sigma)\Lambda_1^3}), \qquad (2.32)$$

where again $\Lambda_1 = (\Lambda^2 m)^{1/3}$ is the scale of the low energy $\mathcal{N}=1$ theory. We will make a series expansion of (2.32) in subsection (3.2) and see that it produces the exact effective superpotential including the Veneziano-Yankielowicz term and the instanton corrections to all order.

2.3 Quartic potential

Finally, let us consider the quartic superpotential

$$W_{\text{tree}} = \frac{1}{2}m\,\text{Tr}\,(\Phi^2) + \frac{1}{4}g\,\text{Tr}\,(\Phi^4). \tag{2.33}$$

In this case, again for the one-cut case in which the gauge symmetry is preserved and all eigenvalues are distributed around the origin, the resolvent is computed noting that $W'(x)^2 - f(x)$ in (2.10) has two double roots and using the asymptotic behavior (2.7),

$$w(x) = -\frac{1}{2S}\Big(gx^3 + mx - (gx^2 + \frac{1}{2}ga^2 + m)\sqrt{x^2 - a^2}\Big), \tag{2.34}$$

where

$$3ga^4 + 4ma^2 - 16S = 0. \tag{2.35}$$

The eigenvalue density follows from (2.34) in (2.13)

$$\rho(\lambda) = \frac{1}{2\pi S}(g\lambda^2 + \frac{1}{2}ga^2 + m)\sqrt{a^2 - \lambda^2}, \tag{2.36}$$

Using (2.33) and (2.36) in (2.15), we obtain the free energy

$$\mathcal{F}_0 = -\frac{1}{2}S^2\log(\frac{a^2}{4}) + \frac{1}{384}(-m^2a^2 + 40mSa^2 + 144S^2). \tag{2.37}$$

Noting from (2.35) that

$$\frac{\partial a}{\partial S} = \frac{4}{3ga^3 + 2ma}, \tag{2.38}$$

the effective superpotential is obtained using (2.37) in (2.16),

$$W_{\text{eff}} = \frac{3}{4}NS - NS\log(\frac{a^2}{4}) - N\frac{(a^2m - 12S)(a^2m - 8S)}{72ga^4 + 48ma^2} + \frac{5ma^2}{48}N, \tag{2.39}$$

where

$$a^2 = \frac{2m}{3g}(\sqrt{1 + 12\frac{gS}{m^2}} - 1). \tag{2.40}$$

We will show in subsection (3.3) that making a series expansion of (2.39) gives the correct leading order Veneziano-Yankielowicz term in addition to all instanton corrections.

3 Integrating in and out

The theory without the tree level deformation is simply pure $\mathcal{N} = 2$ $U(N)$ gauge theory and it has been well studied. The quantum moduli space of this theory can be parameterized by the hyperelliptic curve [1, 13]

$$y^2 = P_N(x)^2 - 4\Lambda^{2N}, \tag{3.1}$$

where y and x are complex coordinated such that

$$P_N(x) = \det(x1 - \Phi) = \prod_{i=1}^{N}(x - x_i), \tag{3.2}$$

x_i being the diagonal elements of Φ, which can be made diagonal using D-flatness conditions. A gauge invariant parametrization of the moduli space can be given by coordinates

$$u_1 = \mathrm{Tr}\,\Phi = \sum_{i=1}^{N} x_i \tag{3.3}$$

and

$$u_p = \frac{1}{p}\mathrm{Tr}\,(\Phi + u_1/N)^p = \frac{1}{p}\sum_{i=1}^{N}(x_i + u_1/N)^p,\ \ p = 2\,\mathrm{to}\,N. \tag{3.4}$$

The quantum moduli space given by (3.1) describes a genus $N-1$ Riemann surface with two types of cycles. One type (β-cycle) is related to the handles of the surface and the second type (α-cycle) connects different handles. A dual scalar field is given by integral of a meromorphic differential over an α-cycle. A dual scalar field vanishes and the associated monopole (or dyon) becomes massless when an α-cycle degenerates. The strong coupling singularities of the quantum moduli space correspond to regions where monopoles become massless. Our interest here is the case where the $U(N)$ gauge symmetry is preserved. This corresponds to the case in which all the α-cycles vanish and the genus $N-1$ Riemann surface degenerates into a sphere. The effective field theory which describes the low energy excitations near these singularities is described in terms of the dual scalar and monopole fields and the exact effective superpotential is given by, following the nonrenormalization and linearity principle of [3],

$$W = \sum_{m=1}^{N-1} A_{D,m}(u)E_m\tilde{E}_m + \sum_{p=2}^{n+1} g_p u_p, \tag{3.5}$$

where E_m and $\tilde{E}_m$ denote the monopole chiral multiplet which becomes massless at the m^{th} singularity and are associated with the dual scalar field $A_{D,m}$. The second term in (3.5) is the tree level deformation. The steps to compute the effective glueball superpotential are integrating out E_m, $\tilde{E}_m$, u_p and u_1 and integrating in S. The equations of motion obtained by minimizing (3.5) with E_m, $\tilde{E}_m$ and u_p are

$$A_{D,m}(u) = 0, \tag{3.6}$$

$$\sum_{m=1}^{N-1} \frac{\partial A_{D,m}(u)}{\partial u_p} E_m\tilde{E}_m + g_p = 0. \tag{3.7}$$

Putting (3.6) in (3.5) gives

$$W_{\mathrm{eff}} = (\sum_{p=2}^{n+1} g_p u_p)|_{A_{D,m}(u)=0}. \tag{3.8}$$

Now (3.7) shows the standard confinement of monopoles and (3.6) expresses the vanishing of the dual scalar fields and the masslessness of the monopoles at the singularities. The effective superpotential is thus given by the tree level deformation evaluated with the constraint that all the dual scalar fields vanish. The glueball effective superpotential is then obtained by integrating out u_1 and integrating in S while at the same time keeping the nonperturbative scale as a parameter in the leading Veneziano-Yankielowicz part of the superpotential. This can be done by introducing an auxiliary field A and minimizing

$$\sum_{p=2}^{n+1}(g_p u_p)|_{A_{D,m}(u)=0}\,(\text{with}\,\Lambda \to A) - 2NS\log(\frac{A}{\Lambda}). \tag{3.9}$$

with A and u_1. Note that when all the α-cycles simultaneously degenerate, $P_N(x)^2 - 4\Lambda^{2N}$ has two single roots and $N-1$ double roots and the hyperelliptic curve factorizes. [14, 6] P_N is then given in terms of the Chebyshev polynomial of the first kind T_N [14],

$$P_N(x) = 2\Lambda^N\,T_N(\frac{x}{2\Lambda}), \tag{3.10}$$

where $T_N(\frac{x}{2\Lambda}) = \cos(N\cos^{-1}(x/(2\Lambda)))$. The diagonal elements x_i of Φ are also parameterized by

$$x_i = 2\Lambda\cos(\frac{i-1/2}{N}\pi). \tag{3.11}$$

Once the effective glueball superpotential is obtained, the free energy in the planar limit is then computed via

$$\mathcal{F}_0 = \frac{1}{N}\int W_{\text{eff}}(S)dS - \frac{1}{2}S^2\log(\Lambda^2) \tag{3.12}$$

with the boundary condition that when there are other than quadratic terms in the tree level potential the free energy reduces to (2.19) at $g_p = 0$ for all $p \neq 2$.

We will start with computing the exact effective superpotentials and the free energies for the quadratic, cubic and quartic tree level deformations. The results exactly agree with the matrix model calculations including the leading Veneziano-Yankielowicz term and all instanton corrections.

3.1 Quadratic deformation

First consider the quadratic deformation with $n = 1$ and $g_2 = m$ in (3.5). In this case, using (3.4) and (3.11) in (3.8) gives

$$W_{\text{eff}} = mN\Lambda^2 + \frac{m}{2N}u_1^2. \tag{3.13}$$

We then integrate in S and integrate out u_1 by minimizing

$$W = mNA^2 + \frac{m}{2N}u_1^2 - 2NS\log(\frac{A}{\Lambda}) \tag{3.14}$$

with A and u_1. This gives

$$W_{\text{eff}} = NS - NS\log(\frac{S}{\Lambda_1^3}), \tag{3.15}$$

where $\Lambda_1 = (m\Lambda^2)^{1/3}$ is the scale of the $\mathcal{N} = 1$ theory related to the scale Λ of the $\mathcal{N} = 2$ theory by threshold matching of the gauge coupling running at energy scale m. (3.15) is exactly the Veneziano-Yankielowicz superpotential. Furthermore, putting (3.15) in (3.12) gives the same free energy (2.19) as that obtained from the matrix model. In other words, the matrix integral in the planar limit of a zero dimensional bosonic matrix model is solved purely using integrating in and out and the Dijkgraaf-Vafa prescription.

3.2 Cubic deformation

Next let us consider the cubic potential with $n = 2$, $g_2 = m$ and $g_3 = g$ in (3.5). In this case, putting (3.4) and (3.11) in (3.8) gives

$$W_{\text{eff}} = (mN + 2gu_1)\Lambda^2 + \frac{m}{2N}u_1^2 + \frac{g}{3N^2}u_1^3. \tag{3.16}$$

Minimizing

$$W = (mN + 2gu_1)A^2 + \frac{m}{2N}u_1^2 + \frac{g}{3N^2}u_1^3 - 2NS\log(\frac{A}{\Lambda}) \tag{3.17}$$

with A and u_1 gives

$$W_{\text{eff}} = NS - NS\log(\frac{S}{\Lambda_1^3}) + NS\log(1 + \frac{2g}{N}u_1) + \frac{m}{2N}u_1^2 + \frac{g}{3N^2}u_1^3\,, \tag{3.18}$$

where

$$u_1 = -\frac{mN}{2g} + \frac{m^2N^2}{6B} + \frac{B}{2g^2}, \tag{3.19}$$

$$B = \left(-4g^5N^3S + \frac{1}{3\sqrt{3}}N^3g^3m^3\sqrt{-1 + 432\frac{g^4S^2}{m^6}}\right)^{1/3} \tag{3.20}$$

and $\Lambda_1 = (m\Lambda^2)^{1/3}$. Expanding (2.32) and (3.18) in g^2S/m^3 gives exactly the same result including the leading Veneziano-Yankielowicz term and the instanton corrections to all order,

$$\begin{aligned} W_{\text{eff}} &= NS - NS\log(\frac{S}{\Lambda_1^3}) - 2NS\frac{g^2S}{m^3} - \frac{32}{3}NS(\frac{g^2S}{m^3})^2 \\ &\quad -\frac{280}{3}NS(\frac{g^2S}{m^3})^3 - 1024NS(\frac{g^2S}{m^3})^4 - NS\mathcal{O}(\frac{g^2S}{m^3})^5. \end{aligned} \tag{3.21}$$

The free energy can also be computed putting (3.18) or (3.21) in (3.12).

3.3 Quartic deformation

Consider the quartic potential with $g_2 = m$, $g_4 = g$ and all other $g_p = 0$ in (3.5). Using (3.4) and (3.11) in (3.8),

$$W_{\text{eff}} = mN\Lambda^2 + \frac{m}{2N}u_1^2 + \frac{3gN}{2}\Lambda^4 + \frac{3g}{N}u_1^2\Lambda^2 + \frac{g}{4N^3}u_1^4. \tag{3.22}$$

Minimizing

$$W = mNA^2 + \frac{m}{2N}u_1^2 + \frac{3gN}{2}A^4 + \frac{3g}{N}u_1^2A^2 + \frac{g}{4N^3}u_1^4 - 2NS\log(\frac{A}{\Lambda}) \tag{3.23}$$

with A and u_1 gives

$$A^2 = \frac{m}{6g}(\sqrt{1+\frac{12gS}{m^2}} - 1), \quad u_1 = 0. \tag{3.24}$$

Putting (3.24) in (3.23), we obtain the effective glueball superpotential

$$W_{\text{eff}} = NS(\frac{1}{2} - \frac{m^2}{12gS}) - NS\log(\frac{\sqrt{1+12gS/m^2}-1}{6\Lambda_1^3 g/m^2}) + N\frac{m^2}{12g}\sqrt{1+\frac{12gS}{m^2}}\,, \tag{3.25}$$

where $\Lambda_1 = (m\Lambda^2)^{1/3}$. Making a series expansion of W_{eff} given by (2.39) and (3.25) again gives exactly the same result,

$$\begin{aligned} W_{\text{eff}} &= NS - NS\log(\frac{S}{\Lambda_1^3}) + \frac{3}{2}NS\frac{gS}{m^2} - \frac{9}{2}NS(\frac{gS}{m^2})^2 \\ &\quad + \frac{45}{2}NS(\frac{gS}{m^2})^3 - \frac{567}{4}NS(\frac{gS}{m^2})^4 + NS\mathcal{O}((\frac{gS}{m^2})^5). \end{aligned} \tag{3.26}$$

Furthermore, putting (3.25) in (3.12) and using the boundary condition discussed below (3.12), the free energy matrix integral in the large N limit is computed using integrating in and out and the result indeed agrees with (2.37).

4 Conclusion

We have shown that the Dijkgraaf-Vafa matrix model prescription with the proper normalization of the partition function and the scheme we presented gives the exact nonperturbative effective superpotential without a need for adding the Veneziano-Yankielowicz term by hand. Both the Dijkgraaf-Vafa matrix model prescription and the integrating in and out procedure give the same result for the full exact nonperturbative effective superpotential including the leading Veneziano-Yankielowicz term and all instanton corrections. It would be important to understand the physical relations between the two quite different structures, one based on the large N limit of zero dimensional bosonic matrix models and the other based on the hypothesis of nonrenormalization and linearity principles. It is interesting to notice that because the parametrization of the fully degenerate Seiberg-Witten hyperelliptic curve was known in terms of the Chebyshev polynomials, the computation of the effective superpotentials on the integrating in side reduced to a simple substitution of the parametrization of the diagonal elements of the adjoint scalar field into the gauge invariant coordinates and integrating in the glueball superfield. Consequently, one-cut matrix integrals are easily computed for any polynomial potential using integrating in and out. It would be interesting to find the polynomials that factorize the Seiberg-Witten curve for more general cases where the degeneration of the quantum moduli space is partial. Once these polynomials are found, doing multi-cut matrix integrals should not in principle be a difficult task using integrating in and out. Integrating in and out can thus be used as an alternative tool to do random matrix integrals.

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